# The First, and Still Best, Method for Non-market Valuation

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## Abstract:

Harold Hotelling proposed the first method for non-market valuation. Most of us believe this is the Travel Cost Method. In this paper I argue that the first method is from Hotelling's seminal study on the extraction of exhaustible resources. The first method is dynamic optimisation as applied to the management of natural resources.

Dynamic optimisation imputes the price of stocks that are neither bought nor sold in a market. Non-market valuation for the environment is usually a question of imputing the value of environmental stocks. Pollutants are almost always stocks. Ecosytem services are flows from environmental stocks. Greenhouse gases, biodiversity, wildlife, national parks, old growth forests, all are stocks. Indeed our ecosystem, is a system of stocks.

Yet our methods for non-market valuation, the Travel Cost Method, Hedonic Pricing, Contingent Valuation, Choice Modelling and the Random Utility Model, are based on static theories of consumer behavior. They do not model environmental stocks, nor include other participants in the economy such as producers and citizens who, in effect, own the ecosystem.

In this paper, I develop a simple but dynamic model of an economy with ecosystem stocks and services. The model shows how the price of ecosystem services can be discovered, provides an example of demand and determines the willingness to accept and willingness to pay for ecosystem services. I conclude that models based on consumer behaviour are misleading. Consumers are complicit with producers in despoiling the environment. Those willing to pay for a healthy ecosystem are citizens.

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## Introduction

The academic year is approaching. Soon, a new mob of excited and eager students will study environmental and natural resource economics in my class. I believe it is my responsibility to present the wisdom of the scribes as objectively and enthusiastically as I can. Natural resource economics—fisheries, forestry, even mining—is easy for me. Environmental economics—particularly nonmarket valuation of the environment—is a challenge. So far I have discussed nonmarket valuation with a straight face. But last year trouble began. I started to crack a smile (Costanza, *et al.*, 1997).

Maybe this year, the exam will include supplementary questions for the more persistent students:

1. List all of the problems with the environment that concern environmental stocks and flows.

2. List all of the nonmarket valuation methods that are derived from a theory of stocks and flows.

I would expect short answers. The marking key might be:

1. 0% "I can't think of any."
   20% "Maybe a few of them."
   40% "Many of them."
   60% "Most of them."
   80% "All of them."
   100% "All of them," plus nuances such as: "Obviously, big P problems are all about stocks and flows but even genetically modified foods are flows that may damage my stock of good health." "Even noise like country western music is the flow from a bad stock of neighbours and reduces the value of my house."

2. 0% "Logic would dictate that all of them are."
   40% "I don't know."
   80% "None of them."
   100% "None of those we studied, but really, this is what natural resource economics and green accounting are all about. Marginal user costs are the prices of resources in their natural state and total user costs are the benefits and costs to society of using natural resources. These are calculated by solving dynamic models of stocks and flows. Perhaps nonmarket valuation could do the same."

On second thought, I probably won't ask these questions this year. Even heretics like myself must teach something. Presenting only those things I believe will leave big gaps in the script. But I agree with the 100% student. Nonmarket valuation of the environment can be based on a dynamic theory of stocks and flows. I argue that it should be.

## A Simple Economy with Ecosystem Stocks and Services

Nonmarket valuation identifies many types of values—use values, including current use and option values for future use, bequest values and existence values. In our choice of names, we recognise stocks and flows. Current use is a flow. Future use, bequests and existence requires stocks. A dynamic model seems the natural method for nonmarket valuation.

Let us suppose our objective is to maximise utility, now and in the future, subject to an economic constraint for the change in our wealth and an ecosystem constraint for the change in ecosystem stocks.

$$J(W_0, S_0) = \max_{Q_1, Q_2} \int_0^T e^{-\rho t} U(Q_1, Q_2) dt + e^{\rho T} V(W_T, S_T)$$

$$\text{subject to:}$$

$$\dot{W} = R(W) - p_1 Q_1 - p_2 Q_2$$

$$\dot{S} = G(S) - Q_2$$

Our lifetime's utility, *J*, depends upon wealth, *W*, and ecosystem stocks, *S*. We maximise utility by choosing commodities, $Q_1$, and the flow of ecosystem services, $Q_2$. These determine our utility of consumption, *U*, which is integrated over time, *t*, and discounted at our consumption rate of time preference, $\rho$. At the end of this life, *T*, we bequeath wealth and ecosystem stocks to future generations and gain utility *V*. Wealth and stocks evolve over time according to differential equations. Wealth increases with revenue, *R*, and decreases with expenditures on commodities, and with the cost of extracting ecosystem services, where $p_1$ and $p_2$ are prices. Ecosystem stocks renew with growth, *G*, and are degraded by the flow of services.

We can already identify some nonmarket values in the model. Current use values are a market value and equal the expenditures needed to extract a flow of ecosystem services. These could be measured by the Travel Cost Method, for example. The bequest value is a utility function and, hence, is unobservable. We can ask for a monetary equivalent using a Contingent Valuation or Choice Modelling survey. Alternately, we can solve the model and use welfare analysis. Future use values are utilities of consumption and are also unobservable. Again, we can ask for monetary equivalents using a survey. Existence values are invisible, lurking in the shadows of model. Current use, future uses, bequests and existence are competing allocations of ecosystem stocks. Efficient allocations are mediated by the price of ecosystem stocks. This price is also lurking in the shadows. To illuminate it, we must solve the model.

Maximising a lifetime's utility is equivalent to maximising a dynamic measure of utility in each time period. This dynamic measure is the Hamilton-Jacobi-Bellman (HJB) equation.

$$-J_t = \max_{Q_1, Q_2} \left[ e^{-\rho t} U + J_W \left[ R(W) - p_1 Q_1 - p_2 Q_2 \right] + J_S \left[ G(S) - Q_2 \right] \right]$$

In this equation $J_t$ is the time derivative of lifetime utility, $J_W$ is the marginal utility of wealth and $J_S$ is the marginal utility of ecosystem stocks. Instead of the HJB equation, we could maximise the Hamiltonian, where the symbol *H* replaces $J_t$, a costate variable, $\lambda$, replaces the marginal utility of wealth and another costate variable, $\psi$, replaces the marginal utility of ecosystem stocks. The HJB equation is presented here because it will prove more convenient for solving the model.

On the right-hand side of the HJB equation, the marginal utility of wealth is multiplied by the change in wealth to get total user benefits or costs. If wealth is increasing, these are total user benefits. If wealth is decreasing, these are total user costs. Similarly, the marginal utility of ecosystem stocks is multiplied by the net quantity used from the ecosystem. If the ecosystem is renewing, the net quantity is positive and these are total user benefits. If the ecosystem is degrading, the net quantity used is negative and these are total user costs. Adding total user benefits and costs to the utility of consumption gives a dynamic measure of utility. Thus, the HJB equation and the Hamiltonian roughly correspond to the concept of Total Economic Value in environmental nonmarket valuation. Here, however, the definition is more precise.

The first-order conditions for commodities and the flow of services are:

$$e^{-\rho t}\,\partial U/\partial Q_1 - J_W\, p_1 = 0$$
$$e^{-\rho t}\,\partial U/\partial Q_2 - J_W\, p_2 - J_S = 0$$

The condition for commodities is similar to conditions from a static model of consumer demand except that the marginal utility of wealth replaces the marginal utility of income. The condition for the flow of services has no counterpart in a static model. It compares marginal utility, marginal extraction costs and the marginal costs of using the ecosystem now instead of conserving for the future. The marginal costs of using the ecosystem are an opportunity cost measured by the marginal utility of ecosystem stocks, $J_S$. Thus, $J_S$ encapsulates our future use and bequest values. As we will see, it is also encapsulates existence values. If there is open access to the ecosystem, $J_S$ will be driven to zero and we will ignore the value of ecosystem stocks in our decisions.

Suppose we applied a static model to the problem of valuing the ecosystem. This would be equivalent to setting $J_S$ to zero. We would assume that future use, bequest and existence values are zero. Could such a model tell us anything at all about the value of the ecosystem?

The first-order conditions for wealth and ecosystem stocks are:

$$\dot{J}_W = -J_W\,\partial R/\partial W$$
$$\dot{J}_S = -J_S\,\partial G/\partial S$$

Marginal utilities of wealth and ecosystem stocks decline over time. The rates of change are the marginal revenue within the economy and marginal growth of the ecosystem. At the end of our life, marginal utilities must equal our discounted marginal bequest values.

$$J_W = e^{-\rho T} V_W$$
$$J_S = e^{-\rho T} V_S$$

Earlier in our life, marginal utilities of wealth and ecosystem stocks include all future marginal revenues and marginal growths.

$$J_W = e^{\int_t^T \frac{\partial R}{\partial W} dt}\, e^{-\rho T} V_W$$
$$J_S = e^{\int_t^T \frac{\partial G}{\partial S} dt}\, e^{-\rho T} V_S$$

However, we can't observe marginal utilities. We would rather know the price of ecosystem stocks in units we can understand such as $/ton. Let us define this price as ratio of the marginal utility of ecosystem stocks to the marginal utility of wealth.

$$\phi = \frac{J_S}{J_W} = e^{\int_t^T \left(\frac{\partial G}{\partial S} - \frac{\partial R}{\partial W}\right) dt} \frac{V_S}{V_W}$$
$$\dot{\phi} = \left(\frac{\partial R}{\partial W} - \frac{\partial G}{\partial S}\right)\phi$$

The nonmarket price of ecosystem stocks is $\phi$. It shows the change in wealth for a change in stocks, $\partial W/\partial S$. In addition to mediating between current and future uses, it and also mediates between holding assets in the economy or in the ecosystem. It grows at a rate equal to marginal revenue minus marginal growth. If marginal revenue in the economy exceeds marginal growth in the ecosystem, the ecosystem will become scarce. Its price will grow exponentially toward infinity. If marginal revenue is less than marginal growth, the ecosystem will become abundant. Its price will go toward zero. Neither zero nor infinity is a realistic price. An economy in equilibrium with the ecosystem will equate marginal revenue with marginal growth and the price of the ecosystem will become constant.

We can use the price of ecosystem stocks to reinterpret the first-order condition for the flow of ecosystem services.

$$e^{-\rho t}\, \partial U/\partial Q_2 - J_W\left(p_2 + \phi\right) = 0$$

The total price we pay for ecosystem services has two components—the price of extracting for current use and the price of depleting ecosystem stocks instead of conserving for the future. If we knew the price of ecosystem stocks, we could solve the first-order conditions as we would any other demand system. Unfortunately, it is still lurking in the shadows and is yet to be illuminated.

## A Dynamic Linear Expenditure System

To shed more light, we can solve a special case of the model to find a dynamic linear expenditure system. For this we need a Stone-Geary utility function.

$$U(Q_1, Q_2) = (Q_1 - \gamma_1)^{\alpha_1}(Q_2 - \gamma_2)^{\alpha_2}$$
$$\frac{\partial U}{\partial Q_1} = \frac{\alpha_1 U}{Q_1 - \gamma_1}$$
$$\frac{\partial U}{\partial Q_2} = \frac{\alpha_2 U}{Q_2 - \gamma_2}$$

Curvature parameters are $\alpha_1$ and $\alpha_2$. Subsistence levels of commodities and ecosystem services are $\gamma_1$ and $\gamma_2$. We can now substitute marginal utilities of consumption into the first-order conditions, add the two conditions together and solve for the marginal utility of wealth.

$$J_W = \frac{e^{-\rho t}\alpha U}{p_1(Q_1-\gamma_1)+(p_2+\phi)(Q_2-\gamma_2)}$$
$$\text{where: } \quad \alpha = \alpha_1 + \alpha_2$$

Then we can substitute the marginal utility of wealth into the first-order condition for ecosystem services and rearrange.

$$(Q_2-\gamma_2) = \frac{\alpha_1}{\alpha_2}\frac{p_1}{p_2+\phi}(Q_1-\gamma_1)$$

This equation looks useful. As economists we argue that any decision about the environment reveals the decision-maker's estimate of nonmarket values. If we knew the $\alpha$ and $\gamma$ parameters, we could follow politicians around and use this equation to calculate how little they value the ecosystem. The parameters could be statistically estimated from survey data or from real data. Notice, however, that information about the flow of ecosystem services is not enough. We must also include information about the consumption of commodities.

For a more compete demand system, we must solve the model. The Appendix derives an analytical solution. Here we interpret it. Recall the HJB equation:

$$-J_t = \max_{Q_1,Q_2}\left[e^{-\rho t}U + J_W\left[R(W) - p_1Q_1 - p_2Q_2\right] + J_S\left[G(S)-Q_2\right]\right]$$

Assume that the utility of consumption, the revenue and the growth functions are:

$$U(Q_1,Q_2) = (Q_1-\gamma_1)^{\alpha_2}(Q_2-\gamma_2)^{\alpha_2}$$
$$R(W) = rW$$
$$G(S) = gS$$
$$r = g$$

Revenue equals the rate of interest, $r$, multiplied by wealth and growth equals the rate of growth, $g$, multiplied by ecosystem stocks. For an economy in equilibrium, the rate of interest equals the rate of growth. A solution is:

$$J(t,W,S) = e^{-\rho t}\left[\frac{\alpha_1}{\alpha_2}\right]^{\alpha_1} p_1^{-\alpha_1}(p_2+\phi)^{-\alpha_2}\left[A(t)\right]^{1-\alpha}\left[B(t,W,S)\right]^{\alpha}$$
$$\text{where:}$$
$$A(t) = \frac{1-\alpha}{\rho-\alpha r}\left(1-\alpha e^{-\frac{\rho-\alpha r}{1-\alpha}(T-t)}\right)$$
$$B(t,W,S) = \left[W + \phi S - \frac{1}{r}(p_1\gamma_1 + p_2\gamma_2)\left(1-be^{-r(T-t)}\right) - \frac{1}{r}\phi\gamma_2\right]$$

The function $A$ is a fancy way of discounting. The function $B$ is more interesting. It contains wealth. Added to this is the existence value of ecosystem stocks defined as the price of stocks multiplied by the quantity. Subtracted from these are the value of subsistence commodities and the value of subsistence flows of ecosystem services. Within functions *A* and *B* are two new parameters, *a* and *b*. These are chosen so that lifetime utility converges to the bequest value at the end of this life when time *t* equals *T*.

$$J(T, W_T, S_T) = e^{-\rho T} V(W_T, S_T)$$

We can now differentiate lifetime utility with respect to wealth to get the marginal utility of wealth. This we can substitute into the first-order conditions for commodities and for the flow of ecosystem services and get a dynamic demand system. In expenditure form, this system is:

$$p_1(Q_1 - \gamma_1) = \frac{\alpha_1}{\alpha_2}\frac{1}{A}\left[W + \phi S - \frac{1}{r}(p_1\gamma_1 + p_2\gamma_2)\left(1 - be^{-r(T-t)}\right) - \frac{1}{r}\phi\gamma_2\right]$$
$$(p_2 + \phi)(Q_2 - \gamma_2) = \frac{1}{A}\left[W + \phi S - \frac{1}{r}(p_1\gamma_1 + p_2\gamma_2)\left(1 - be^{-r(T-t)}\right) - \frac{1}{r}\phi\gamma_2\right]$$

Dynamic demands differ from static demands. They are functions of wealth and the existence value of stocks rather than income. Unfortunately, we don't know the existence value because we don't know the price of ecosystem stocks. We were hoping the model would impute it for us. Instead, we must find some data and statistically estimate the expenditure equations. This is similar in spirit to Choice Modelling and Random Utility methods. In this method, however, the price of ecosystem stocks, $\phi$, is a parameter that can be estimated directly.

Finally, we can derive the willingness to accept and the willingness to pay for ecosystem services. The willingness to accept is a compensating variation, *C*. A compensating variation is the minimum amount of money people must receive to voluntarily accept a price for ecosystem services. The willingness to pay is an equivalent variation, *E*. An equivalent variation is the maximum amount people would pay to avoid paying a price for ecosystem services.

$$E = \left(1 - \left(\frac{p_2}{p_2 + \phi}\right)^{\frac{\alpha_2}{\alpha}}\right)\left[W - \frac{1}{r}(p_1\gamma_1 + p_2\gamma_2)\left(1 - Be^{-r(T-t)}\right)\right] - \phi\left(\frac{p_2}{p_2 + \phi}\right)^{\frac{\alpha_2}{\alpha}}\left(S - \frac{1}{r}\gamma_2\right)$$

$$C = \left(\frac{p_2 + \phi}{p_2}\right)^{\frac{\alpha_2}{\alpha}} E$$

In this model, people are both consumers and owners of the ecosystem, which makes their compensating and equivalent variations ambiguous. Their variations can be positive, negative or zero. Still, these equations might be quite handy. Unlike the demand system, there are no quantities for commodities or for the flow of ecosystem services. There are only prices, wealth and ecosystem stocks. Similar to a Contingent Valuation survey, we could ask willingness to pay questions and use the results to statistically estimate the price of ecosystem stocks.

Let's investigate why variations can be either positive or negative. Suppose people are only consumers, which eliminates the last term from the equivalent variation.

$$E = \left(1 - \left(\frac{p_2}{p_2 + \phi}\right)^{\frac{\alpha_2}{\alpha}}\right)\left[W - \frac{1}{r}\left(p_1\gamma_1 + p_2\gamma_2\right)\left(1 - Be^{-r(T-t)}\right)\right]$$

For consumers, the variations are always positive. Consumers require a positive compensation to accept a price for ecosystem services and will pay a positive lump sum to avoid a price for services. The compensating variation exceeds the equivalent variation, possibly by a large amount. If the price of ecosystem services is large, the compensating variation can be two or three times larger than the equivalent variation.

Now suppose people are citizens who act as owners of the ecosystem but are not consumers. This eliminates the first term from the equivalent variation.

$$E = -\phi\left(\frac{p_2}{p_2 + \phi}\right)^{\frac{\alpha_2}{\alpha}}\left(S - \frac{1}{r}\gamma_2\right)$$

For owners, the variations are always negative. Owners prefer a price for ecosystem services. The price gives the ecosystem its future use, bequest and existence values. The compensating variation is also negative, but larger in magnitude than the equivalent variation.

Of course, people are both consumers and owners. Their willingness to pay can be positive, negative or zero. In this model, a positive willingness to pay denotes a consumer; a negative willingness to pay denotes an owner. What does this mean for nonmarket valuation? Our surveys routinely include sections to identify protestors. These are people who believe in conserving the ecosystem but are not willing to pay a positive amount. Are these people acting as if they own the ecosystem? Are these the people we should be surveying instead consumers?

## Conclusions

Since the days of Hotelling, natural resource economists have modelled dynamic systems. Usually they study production over time using natural resources as inputs. Often they ignore the consequences to the larger environment. Environmental economists rarely model systems. Usually they study consumption, as if the environment were a shopping basket of commodities. Often they ignore the consequences to the larger economy. Ecological economists criticise the fragmentation in our discipline and advocate holistic studies of the economy and ecosystem. Intriguingly, Hotelling was both a natural resource and an environmental economist. Perhaps he would have been an ecological economist as well.

Perhaps it is time to combine the best of natural resource and environmental economics. Then, perhaps, nonmarket valuation will become less controversial. We all agree that nonmarket valuation is crucial for better management of the ecosystem. We disagree, however, on the methods. How can future use values, bequest values and existence values be measured using a static model? Why do we exclude owners and base our studies on theories of consumer behaviour? As every natural resource economist knows, consumers are exploiters; owners are citizens.

Soon, perhaps, there will be fewer gaps in my script.

## Reference

Robert Costanza, Ralph d'Arge, Rudolf de Groot, Stephen Farber, Monica Grasso, Bruce Hannon, Karin Limburg, Shahid Naeem, Robert V. Oneill, Jose Paruelo, Robert G. Raskin, Paul Sutton and Marjan van den Belt, "The value of the world's ecosystem services and natural capital," *Nature*: Vol 387, 15 May 1997.

## Appendix

Here we derive the analytical solution. You may find this fascinating, or you may not. Recall the simple model of ecosystem services.

$$J(W_0, S_0) = \max_{Q_1, Q_2} \int_0^T e^{-\rho t} U(Q_1, Q_2) dt + e^{\rho T} V(W_T, S_T)$$

$$\text{subject to:}$$

$$\dot{W} = R(W) - p_1 Q_1 - p_2 Q_2$$

$$\dot{S} = G(S) - Q_2$$

Also recall the Hamilton-Jacobi-Bellman (HJB) equation in each time period.

$$-J_t = \max_{Q_1, Q_2} \left[ e^{-\rho t} U + J_W [R(W) - p_1 Q_1 - p_2 Q_2] + J_S [G(S) - Q_2] \right]$$

Assume functional forms for revenue and growth.

$$R(W) = rW$$

$$G(S) = gS$$

The first-order conditions are:

$$e^{-\rho t} \partial U / \partial Q_1 - J_W p_1 = 0$$

$$e^{-\rho t} \partial U / \partial Q_2 - J_W p_2 - J_S = 0$$

$$\dot{J}_W = -r J_W$$

$$\dot{J}_S = -g J_S$$

Use these and transversality conditions for marginal utilities to find the nonmarket price for ecosystem stocks.

$$
\begin{aligned}
J_W &= e^{-rt}e^{-\rho T}V_W \\
J_S &= e^{-gt}e^{-\rho T}V_W \\
\phi &= \frac{J_W}{J_S} = e^{(r-g)t}\frac{V_S}{V_W} \\
\dot{\phi} &= (r-g)\phi
\end{aligned}
$$

The nonmarket price is $\phi$. It grows at a rate equal to the interest rate, $r$, minus the renewal rate, $g$. Reformulate the HJB equation using this price.

$$
-J_t = \max_{Q_1,Q_2}\left[e^{-\rho t}U + J_W\left[rW + g\phi S - p_1Q_1 - (p_2+\phi)Q_2\right]\right]
$$

Again derive the first-order conditions.

$$
\begin{aligned}
& e^{-\rho t}\,\partial U/\partial Q_1 - J_W\,p_1 = 0 \\
& e^{-\rho t}\,\partial U/\partial Q_2 - J_W\,(p_2+\phi) = 0 \\
& \dot{J}_W = -rJ_W \\
& \dot{J}_W\phi + J_W\dot{\phi} = -gJ_W\phi
\end{aligned}
$$

To progress further, we can specify a Stone-Geary utility function for consumption.

$$
\begin{aligned}
U(Q_1,Q_2) &= (Q_1-\gamma_1)^{\alpha_1}(Q_2-\gamma_2)^{\alpha_2} \\
\frac{\partial U}{\partial Q_1} &= \frac{\alpha_1 U}{Q_1-\gamma_1} \\
\frac{\partial U}{\partial Q_2} &= \frac{\alpha_2 U}{Q_2-\gamma_2}
\end{aligned}
$$

Add the first-order conditions together and solve for the marginal utility of wealth.

$$
J_W = \frac{e^{-\rho t}\alpha U}{p_1(Q_1-\gamma_1)+(p_2+\phi)(Q_2-\gamma_2)}
$$
$$
\text{where}: \quad \alpha = \alpha_1 + \alpha_2
$$

Substitute the marginal utility of wealth into the first-order condition for the flow of ecosystem services and solve.

$$
(Q_2-\gamma_2) = \frac{\alpha_1}{\alpha_2}\frac{p_1}{p_2+\phi}(Q_1-\gamma_1)
$$

Eliminate the flow of ecosystem services from utility.

$$U(Q_1, Q_2) = \pi(Q_1 - \gamma_1)^{\alpha}$$
$$\text{where}: \quad \pi = \frac{\alpha_2}{\alpha_1}\frac{p_1}{p_2 + \phi}$$

Reformulate the HJB equation yet again.

$$-J_t = \max_{Q_1}\left[e^{-\rho t}\pi^{\alpha_2}(Q_1 - \gamma_1)^{\alpha} + J_W\left[rW + g\phi S - p_1\gamma_1 - (p_2 + \phi)(\pi(Q_1 - \gamma_1) + \gamma_2)\right]\right]$$

From this version of the HJB equation, find the first-order condition for commodities and solve.

$$e^{-\rho t}\pi^{\alpha_2}\alpha(Q_1 - \gamma_1)^{\alpha-1} - J_W(p_2 + \phi)\pi = 0$$
$$(Q_1 - \gamma_1) = \left[J_W e^{\rho t}\pi^{1-\alpha_2}\frac{(p_2 + \phi)}{\alpha}\right]^{\frac{1}{\alpha-1}}$$

Finally substitute commodities into the HJB equation and simplify to get the maximised HJB equation.

$$-J_t = (1-\alpha)e^{-\rho t}\pi^{\alpha_2}\left[J_W e^{\rho t}\pi^{1-\alpha_2}\frac{p_2 + \phi}{\alpha}\right]^{\frac{\alpha}{\alpha-1}} + J_W\left[rW + g\phi S - p_1\gamma_1 - (p_2 + \phi)\gamma_2\right]$$

With this simple model, we can integrate the HJB equation to find utility. Unfortunately, integrating is not easy. Checking a trial solution is easier, if tedious. Try:

$$J(t, W, S) = e^{-\rho t}\Omega(t)[A(t)]^{1-\alpha}[B(t, W, S)]^{\alpha}$$

In this trial solution, $\Omega$ will contain prices and parameters, $A$ will be a function of time and $B$ will be a function of time, wealth and ecosystem stocks. Evaluate the derivatives:

$$J_t = -\rho J + \frac{\Omega_t}{\Omega}J + (1-\alpha)\frac{A_t}{A}J + \alpha\frac{B_t}{B}J$$
$$J_W = \alpha e^{-\rho t}\Omega A^{1-\alpha}B^{\alpha-1}B_W = \alpha\frac{B_W}{B}J$$
$$J_S = \alpha\frac{B_s}{B}J$$

Next, evaluate the first term on the right-hand side of the HJB equation. Substitute in the marginal utility of wealth and simplify.

$$(1-\alpha)e^{-\rho t}\pi^{\alpha_2}\left[J_W e^{\rho t}\pi^{1-\alpha_2}\frac{p_2 + \phi}{\alpha}\right]^{\frac{\alpha}{\alpha-1}} = (1-\alpha)\pi^{\frac{\alpha_2}{1-\alpha}}\Omega^{\frac{1}{\alpha-1}}\left[\frac{\alpha_2}{\alpha_1}B_W p_1\right]^{\frac{\alpha}{\alpha-1}}\frac{1}{A}J$$

Well maybe it's not so simple. The HJB equation becomes:

$$0 = -\rho J + \frac{\Omega_t}{\Omega} J + (1-\alpha)\frac{A_t}{A} J + \alpha \frac{B_t}{B} J$$
$$+ (1-\alpha)\pi^{\frac{\alpha_2}{1-\alpha}} \Omega^{\frac{1}{\alpha-1}} \left[\frac{\alpha_2}{\alpha_1} B_W p_1\right]^{\frac{\alpha}{\alpha-1}} \frac{1}{A} J$$
$$+ \alpha \frac{B_W}{B}\left[rW + g\phi S - p_1\gamma_1 - (p_2+\phi)\gamma_2\right] J$$

This is still a mess, so impose the constraint:

$$\pi^{\frac{\alpha_2}{1-\alpha}} \Omega^{\frac{1}{\alpha-1}} \left[\frac{\alpha_2}{\alpha_1} B_W p_1\right]^{\frac{\alpha}{\alpha-1}} = 1$$
$$\text{where :}$$
$$\Omega = \left[\frac{\alpha_1}{\alpha_2}\right]^{\alpha_1} p_1^{-\alpha_1} (p_2+\phi)^{-\alpha_2}$$
$$B_W = 1$$

Simplify further and rearrange the HJB equation.

$$0 = -\rho + \frac{\Omega_t}{\Omega} + \frac{(1-\alpha)}{A}[A_t + 1] + \frac{\alpha}{B}\left[B_t + B_W\left[rW + g\phi S - p_1\gamma_1 - (p_2+\phi)\gamma_2\right]\right]$$

This equation is a bit less challenging, so let's go for *A*. Try:

$$A(t) = \frac{1-\alpha}{\rho - \alpha r}\left(1 - ae^{\frac{\rho-\alpha r}{1-\alpha}(T-t)}\right)$$
$$A_t = -ae^{\frac{\rho-\alpha r}{1-\alpha}(T-t)} = \frac{\rho-\alpha r}{1-\alpha} A - 1$$
$$\frac{1-\alpha}{A}[A_t + 1] = \rho - \alpha r$$

This result looks promising. Now try a solution for *B*:

$$B(t,W,S) = \left[W + \phi S - \frac{1}{r}(p_1\gamma_1 + p_2\gamma_2)\left(1 - be^{-r(T-t)}\right) - \frac{1}{g}\phi\gamma_2\right]$$
$$B_t = \left[(r-g)\phi S + (p_1\gamma_1 + p_2\gamma_2)be^{-r(T-t)} - \frac{r-g}{g}\phi\gamma_2\right]$$
$$\frac{\alpha}{B}\left[B_t + \left[rW + g\phi S - p_1\gamma_1 - (p_2+\phi)\gamma_2\right]\right] = \alpha r$$

With these forms of *A* and *B*, the HJB equation is:

$$-\rho + \frac{\Omega_t}{\Omega} + \rho - \alpha r + \alpha r = \frac{\Omega_t}{\Omega}$$

Unfortunately,

$$\frac{\Omega_t}{\Omega} = -\alpha_2 (r - g) \frac{\phi}{p_2 + \phi}$$

Bummer. After many attempts, I gave up trying to eliminate the growth in the price of ecosystem stocks and assumed that:

$$r = g$$

Now we have and analytical solution. Substitute the marginal utility of wealth into the first-order conditions for commodities and the flow of ecosystem services.

$$p_1 (Q_1 - \gamma_1) = \frac{\alpha_1}{\alpha_2} \frac{1}{A} \left[ W + \phi S - \frac{1}{r} (p_1 \gamma_1 + p_2 \gamma_2)\left(1 - be^{-r(T-t)}\right) - \frac{1}{r} \phi \gamma_2 \right]$$
$$(p_2 + \phi)(Q_2 - \gamma_2) = \frac{1}{A} \left[ W + \phi S - \frac{1}{r} (p_1 \gamma_1 + p_2 \gamma_2)\left(1 - be^{-r(T-t)}\right) - \frac{1}{r} \phi \gamma_2 \right]$$

Finally, we can derive the willingness to accept and the willingness to pay for ecosystem services. We calculate these by equating lifetime utilities for two different scenarios. First define the willingness to accept.

$$e^{-\rho t} \left[ \frac{\alpha_1}{\alpha_2} \right]^{\alpha_1} p_1^{-\alpha_1} p_2^{-\alpha_2} A^{1-\alpha} \left[ W - \frac{1}{r} (p_1 \gamma_1 + p_2 \gamma_2)\left(1 - Be^{-r(T-t)}\right) \right]^{\alpha}$$
$$= e^{-\rho t} \left[ \frac{\alpha_1}{\alpha_2} \right]^{\alpha_1} p_1^{-\alpha_1} (p_2 + \phi)^{-\alpha_2} A^{1-\alpha} \left[ W + C + \phi S - \frac{1}{r} (p_1 \gamma_1 + p_2 \gamma_2)\left(1 - Be^{-r(T-t)}\right) - \frac{1}{r} \phi \gamma_2 \right]^{\alpha}$$

On the left-hand side is lifetime utility with a zero price for ecosystem stocks. On the right-hand side is lifetime utility with a positive price and a compensating variation, *C*. Next define the willingness to pay.

$$e^{-\rho t} \left[ \frac{\alpha_1}{\alpha_2} \right]^{\alpha_1} p_1^{-\alpha_1} p_2^{-\alpha_2} A^{1-\alpha} \left[ W - E - \frac{1}{r} (p_1 \gamma_1 + p_2 \gamma_2)\left(1 - Be^{-r(T-t)}\right) \right]^{\alpha}$$
$$= e^{-\rho t} \left[ \frac{\alpha_1}{\alpha_2} \right]^{\alpha_1} p_1^{-\alpha_1} (p_2 + \phi)^{-\alpha_2} A^{1-\alpha} \left[ W + \phi S - \frac{1}{r} (p_1 \gamma_1 + p_2 \gamma_2)\left(1 - Be^{-r(T-t)}\right) - \frac{1}{r} \phi \gamma_2 \right]^{\alpha}$$

On the left hand-side is lifetime utility with a zero price for ecosystem stocks and an equivalent variation, *E*. Lots of terms cancel and we can solve for the compensating and equivalent variations.

$$C = \left( \left( \frac{p_2 + \phi}{p_2} \right)^{\frac{\alpha_2}{\alpha}} - 1 \right) \left[ W - \frac{1}{r} \left( p_1 \gamma_1 + p_2 \gamma_2 \right) \left( 1 - Be^{-r(T-t)} \right) \right] - \phi \left( S - \frac{1}{r} \gamma_2 \right)$$

$$E = \left( 1 - \left( \frac{p_2}{p_2 + \phi} \right)^{\frac{\alpha_2}{\alpha}} \right) \left[ W - \frac{1}{r} \left( p_1 \gamma_1 + p_2 \gamma_2 \right) \left( 1 - Be^{-r(T-t)} \right) \right] - \phi \left( \frac{p_2}{p_2 + \phi} \right)^{\frac{\alpha_2}{\alpha}} \left( S - \frac{1}{r} \gamma_2 \right)$$

That was fun.